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Like-linearizations of vector fields ☆

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Abstract

We characterize the n-dimensional vector fields (with or without null linear parts) which can be transformed, under conjugation or orbital equivalence, into their quasi-homogeneous parts of minimum degree and, therefore, have the same dynamics. We give several examples of nilpotent and degenerate systems.

1. Introduction

One of the goals in the analysis of dynamic systems is to give a complete characterization of the geometry of the orbital structure near a fixed point. Usually, a method for its determination lies in transforming the system into another whose orbital structure is known. This way, one technique is to express the vector field as a perturbation of another given, and to analyze when both vector fields have the same dynamic. So far, this method has only been used for vector fields with non-null linear part.

A classical problem asks whether a vector field can be transformed, by means of a near-identity change of variables into a linear vector field (in such a case, it says that the vector field is linearizable). It is known that the orbital structure near a hyperbolic singular point (the eigenvalues of the linear approximation at a singular point has nonzero real part) is qualitatively the same as the orbital structure given by the associated linear dynamical system. Nevertheless, in general, the diffeomorphism is not analytic in an open neighbourhood of the singular point.

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At most, $\mathcal{C}^2$ conjugation can be guaranteed, see Hartman [14] and Grobman [13]. This way, as a consequence of the works of Poincaré [17] and Chen [10], if the eigenvalues of the matrix are non-resonant, the vector field is linearizable under smooth conjugation. If, moreover, the eigenvalues $\lambda_1, \dots, \lambda_n$ of the linear approximation belong to the Poincaré domain, i.e. if the convex hull of the points $\lambda_1, \dots, \lambda_n$ in the complex plane does not contain zero, then there is a convergent normalizing transformation (the vector field is analytically conjugated to its linear part), Poincaré [17]. Other conditions of convergence of transformations to its linear part for a vector field with a hyperbolic singular point were given by Siegel, see Arnold [5].

Also, we point out the works of Bruno about convergent normalizing transformations (i.e., transformations that bring a vector field to a Poincaré–Dulac normal form). In fact, he proves that if a Poincaré–Dulac normal form of a vector field verifies the condition "A" and the linear part verifies the condition "ω", then there exists a convergent normalizing transformation, see [7]. As a consequence, he proves that if a singular point is a center of a system with linear part $-y\partial_x + x\partial_y$ then there exists a convergent normalizing transformation, whereas if the singular point is a weak focus, we cannot, in general, guarantee the existence of such a transformation.

In recent years, both problems, linearization and its convergence, have been analyzed by means of the existence of symmetries in a neighbourhood of the singular point; we can choose the origin as the singular point. We recall that a vector field $\mathbf{F}$ admits a nontrivial symmetry if there is a vector field $\mathbf{G}$, transversal to $\mathbf{F}$ at the origin, such that the Lie bracket of both fields, $[\mathbf{F}, \mathbf{G}] = (D\mathbf{F})\mathbf{G} - (D\mathbf{G})\mathbf{F}$, is null. In such a case, it says that $\mathbf{F}$ and $\mathbf{G}$ commute, and $\mathbf{G}$ is a commutator of $\mathbf{F}$.

Concerning the convergence problem, for the two-dimensional systems with non-null linear part, it follows from results of Markhashov [16] and Bruno and Walcher [8], that there is a convergent transformation to Poincaré–Dulac normal form if and only if there exists a nontrivial symmetry. Cicogna [9] extends parts of this result to a higher dimension.

We emphasize that, in general, the existence of a commutator of the vector field does not ensure that the vector field is linearizable. For instance, the systems

$$\begin{cases} \dot{x} = -y + xP_{2l}(x, y)\displaystyle\sum_{j=0}^{r} a_j\left(x^2 + y^2\right)^j, \\ \dot{y} = x + yP_{2l}(x, y)\displaystyle\sum_{j=0}^{r} a_j\left(x^2 + y^2\right)^j, \end{cases}$$

with $P_{2l}(x, y)$ homogeneous polynomial of degree $2l$, $l \geqslant 0$, and a_j, $j = 0, \dots, r$, arbitrary real numbers, have a convergent normalizing transformation, since they commute with

$$\left(x\sum_{j=0}^{r} a_j\left(x^2 + y^2\right)^{j+l}, y\sum_{j=0}^{r} a_j\left(x^2 + y^2\right)^{j+l}\right)^T.$$

Nevertheless, there are vector fields of this family whose origin is a focus and therefore they are not linearizable. Algaba and Reyes [4] characterize the linearizable two-dimensional vector fields of type center-focus through the existence of commutators with null linear part.

On the other hand, in relation to the linearization problem, Bambusi et al. [6] characterized, at least theoretically, the linearizable vector fields.

Theorem 1.1 *(Bambusi, Cicogna, Gaeta and Marmo). Let* $\mathbf{F}$ *be a formal (analytic) vector field in* $\mathbb{R}^n$. $\mathbf{F}$ *is linearizable by a formal (analytic) near-identity change of coordinates if and only* *if the equation* $[\mathbf{F},\mathbf{G}]=\mathbf{0}$ *admits a solution for which* $(D\mathbf{G})(\mathbf{0})=I$. *Moreover, in such a case, there exists a formal* (*analytic*) *change of coordinates which linearizes to* $\mathbf{F}$ *and also linearizes to* $\mathbf{G}$.

Obviously, the difference between the results of Bruno and Walcher [8] and Bambusi et al. [6] is that the commutator either has or does not have null linear part.

This equivalence between linearization and commutation is well known for the analytic two-dimensional vector fields with a singular point of type center-focus, i.e. the vector fields whose linear part is $-y\partial_x+x\partial_y$. Concretely, a system type center-focus is linearizable under analytic conjugation if and only if it has a commutator with linear part $x\partial_x+y\partial_y$; in this case, the center is an isochronous center, see Algaba et al. [1].

We recall that a vector field $\mathbf{G}$ is a normalizator of a vector field $\mathbf{F}$ if $[\mathbf{F},\mathbf{G}]=\mu\mathbf{F}$ where μ is a scalar function. For center, in [1] the link between the linearization under analytic orbital equivalence and the existence of an analytic normalizator of the vector field with linear part $x\partial_x+y\partial_y$ is also proved. Giné and Grau [12] prove a similar result for two-dimensional vector fields whose origin is a nondegenerate singular point (that is, the determinant of the linear part at origin is non-zero).

In our work, we deal with vector fields whose linear part at origin can be null. In order to express our results, we have to recall some definitions and concepts.

Definition 1. Given $\mathbf{t}=(t_1,\dots,t_n)\in\mathbb{N}^n$, a function f is a $\mathbf{t}$-homogeneous function (also called quasi-homogeneous function with respect to the type $\mathbf{t}$) of degree k, with k non-negative integer, if $f(\varepsilon^{t_1}x_1,\dots,\varepsilon^{t_n}x_n)=\varepsilon^k f(x_1,\dots,x_n)$. We will denote by $\mathcal{P}_k^{\mathbf{t}}$ the vector space of $\mathbf{t}$-homogeneous polynomials of degree k. A vector field $\mathbf{F}=F_1\partial_{x_1}+\cdots+F_n\partial_{x_n}$ is $\mathbf{t}$-homogeneous of degree k, with k integer number, if $F_i\in\mathcal{P}_{k+t_i}^{\mathbf{t}}$, $i=1,\dots,n$. The vector space of $\mathbf{t}$-homogeneous polynomial vector fields of degree k will be denoted by $\mathcal{Q}_k^{\mathbf{t}}$.

Throughout this paper by smooth we mean $\mathcal{C}^\infty$. Given a type $\mathbf{t}\in\mathbb{N}^n$, each formal vector field $\mathbf{F}$, with $\mathbf{F}(\mathbf{0})=\mathbf{0}$, can be written as $\mathbf{F}=\sum_{j\geqslant r}\mathbf{F}_j$ where $\mathbf{F}_j\in\mathcal{Q}_j^{\mathbf{t}}$. In some sense, $\mathbf{F}$ can be understood as a perturbation of $\mathbf{F}_r$ with higher-degree $\mathbf{t}$-homogeneous terms. Thus, for any $M\in\mathbb{N}\cup\{\infty\}$, we can define the $\mathbf{t}$-homogeneous M-jet of a smooth vector field F as $\mathcal{J}_{\mathbf{t}}^M\mathbf{F}=\sum_{j=r}^M\mathbf{F}_j$ with $\mathbf{F}_j\in\mathcal{Q}_j^{\mathbf{t}}$. So $\mathbf{F}-\mathcal{J}_{\mathbf{t}}^\infty\mathbf{F}$ is a flat vector field at origin. In our work, we give necessary and sufficient conditions so that a vector field has the same orbital structure as a quasi-homogeneous vector field, which is non-linear, in general. Concretely, given a type $\mathbf{t}$, we characterize the vector fields which are formally, or smoothly, or analytically conjugated to their $\mathbf{t}$-homogeneous part of minimum degree. As a consequence, it has the result given by Bambusi et al., for $\mathbf{t}=(1,\dots,1)$. Analogously, we generalize to a higher dimension the result given by Giné and Grau for smooth and analytic orbital equivalence.

For an analysis of the quasi-homogeneous vector fields, see [2,3] and references therein.

Our contribution is the following: the existence of a vector field normalizator of $\mathbf{F}$ allows us to remove the terms of degree greater than r in order to study the dynamic of $\mathbf{F}$, under conjugation (Theorem 1.2) and under orbital equivalence (Theorem 1.3).

In what follows, we suppose a type $\mathbf{t}=(t_1,\dots,t_n)$ to be fixed, and we denote by $\mathbf{D}_0$, the vector field $\mathbf{D}_0=t_1\partial_{x_1}+\cdots+t_n\partial_{x_n}\in\mathcal{Q}_0^{\mathbf{t}}$. Throughout this paper, the Lie derivative of a scalar function λ of class $\mathcal{C}^1$ by $\mathbf{F}=F_1\partial_{x_1}+\cdots+F_n\partial_{x_n}$ is defined by $L_{\mathbf{F}}\lambda:=\frac{\partial\lambda}{\partial x_1}F_1+\cdots+\frac{\partial\lambda}{\partial x_n}F_n$.

Theorem 1.2 *(Like-linearization under conjugation). Let* $\mathbf{F}$ *be a smooth* (*analytic*) *vector field with* $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ *and* $\mathcal{J}_{\mathbf{t}}^{r}\mathbf{F} = \mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. *Then,* $\mathbf{F}$ *and* $\mathbf{F}_r$ *are smoothly* (*analytically*) *conjugated if and only if there exists a smooth* (*analytic*) *vector field* $\mathbf{G}$, *with* $\mathcal{J}_{\mathbf{t}}^{0}\mathbf{G} = \mathbf{D}_0$, *such that* $[\mathbf{F}, \mathbf{G}] = r\mathbf{F}$.

Moreover, in such a case, there is a smooth (*analytic*) *near-identity nonlinear change of coordinates which transforms* $\mathbf{F}$ *into* $\mathbf{F}_r$ *and also linearizes to* $\mathbf{G}$.

Theorem 1.3 *(Like-linearization under orbital equivalence). Let* $\mathbf{F}$ *be a smooth* (*analytic*) *vector field with* $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ *and* $\mathcal{J}_{\mathbf{t}}^{r}\mathbf{F} = \mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$. *Then,* $\mathbf{F}$ *and* $\mathbf{F}_r$ *are smoothly* (*analytically*) *orbital equivalents if and only if there exists a smooth* (*analytic*) *vector field* $\mathbf{G}$, *with* $\mathcal{J}_{\mathbf{t}}^{0}\mathbf{G} = \mathbf{D}_0$ *and a smooth* (*analytic*) *scalar function* μ *with* $\mu(\mathbf{0}) = r$ *verifying* $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$.

Moreover, in such a case, $\mathbf{F}$ *is smoothly* (*analytically*) *conjugated to* $(1+\lambda)\mathbf{F}$ *being* λ *a smooth* (*analytic*) *scalar function with* $\lambda(\mathbf{0}) = 0$ *such that* $L_{\mathbf{G}}\lambda = (1+\lambda)(r-\mu)$.

If we take $\mathbf{t} = (1, \ldots, 1)$ and $r = 0$, $\mathbf{F}_0 + \mathbf{F}_1 + \cdots$ is the expansion into homogeneous polynomial vector fields of $\mathbf{F}$, being $\mathbf{F}_0$ its linear part. So, as particular cases of Theorems 1.2 and 1.3 we obtain the following results (Corollaries 1.1 and 1.2) which are the main results of the papers [12] and [1], respectively.

Corollary 1.1. *Let* $\mathbf{F}$ *be a smooth* (*analytic*) *vector field. It holds*:

(i) $\mathbf{F}$ *is linearizable under smooth* (*analytic*) *conjugation if and only if there exists a smooth* (*analytic*) *vector field* $\mathbf{G}$ *with* $\mathbf{G}(\mathbf{x}) = \sum_{i=1}^{n} x_i \partial_{x_i} + \mathcal{O}(|\mathbf{x}|^2)$ *such that* $[\mathbf{F}, \mathbf{G}] = \mathbf{0}$.

(ii) $\mathbf{F}$ *is linearizable under smooth* (*analytic*) *orbital equivalence if and only if there exists a smooth* (*analytic*) *vector field* $\mathbf{G}$ *with* $\mathbf{G}(\mathbf{x}) = \sum_{i=1}^{n} x_i \partial_{x_i} + \mathcal{O}(|\mathbf{x}|^2)$ *and a smooth* (*analytic*) *scalar function* μ *with* $\mu(\mathbf{0}) = 0$ *such that* $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$.

Corollary 1.2. *If* $\mathbf{F}$ *is an analytic two-dimensional vector field type center-focus, i.e.* $\mathbf{F}(x, y) = -y\partial_x + x\partial_y + \mathcal{O}(2)$ *it has*:

(i) *the origin is an isochronous center* (*all the closed orbits neighbouring* O *have the same period*) *of* $\mathbf{F}$ *if and only if there is an analytic vector field* $\mathbf{G}$ *with* $\mathbf{G}(x, y) = x\partial_x + y\partial_y + \mathcal{O}(2)$ *such that* $[\mathbf{F}, \mathbf{G}] = \mathbf{0}$,

(ii) *the origin is a center of* $\mathbf{F}$ *if and only if there is an analytic vector field* $\mathbf{G}$ *with* $\mathbf{G}(x, y) = x\partial_x + y\partial_y + \mathcal{O}(2)$ *and an analytic scalar function* μ *with* $\mu(\mathbf{0}) = 0$ *such that* $[\mathbf{F}, \mathbf{G}] = \mu\mathbf{F}$.

The remainder of the paper is organized as follows. In Section 2, we prove the main results. In Section 3, we apply our results to several examples.

2. Proof of the main results

We cite some properties of the quasi-homogeneous polynomials and vector fields which are easily obtained.

Lemma 1. *The following properties hold*:

1. *If* $\lambda \in \mathcal{P}_i^{\mathbf{t}}$ *and* $\mathbf{F} \in \mathcal{Q}_j^{\mathbf{t}}$, *then* $L_{\mathbf{F}}\lambda \in \mathcal{P}_{i+j}^{\mathbf{t}}$.
2. *If* $\lambda \in \mathcal{P}_i^{\mathbf{t}}$, *then* $L_{\mathbf{D}_0}\lambda = i\lambda$ (*Euler Theorem for quasi-homogeneous functions*).
3. *If* $\mathbf{F} \in \mathcal{Q}_i^{\mathbf{t}}$, *then* $[\mathbf{F}, \mathbf{D}_0] = i\mathbf{F}$.

In general, fixed a type $\mathbf{t}$, the expansion into quasi-homogeneous terms of a near-identity change of variables has $\mathbf{t}$-homogeneous terms whose degree is negative. For instance, the expansion into $(1,3)$-homogeneous terms of $\varphi(x,y)=(x,y)^T+(\sum_{i+j\geqslant 2} p_{ij}x^iy^j, \sum_{i+j\geqslant 2} q_{ij}x^iy^j)^T$ is

$$\varphi(x,y)=\underbrace{\begin{pmatrix}0\\ q_{20}x^2\end{pmatrix}}_{\varphi_{-1}}+\underbrace{\begin{pmatrix}x\\ q_{30}x^3+y\end{pmatrix}}_{\varphi_0}+\underbrace{\begin{pmatrix}p_{20}x^2\\ q_{11}xy+q_{40}x^4\end{pmatrix}}_{\varphi_1}+\cdots.$$

The following result allows us to consider only near-identity changes of variables with $\mathbf{t}$-homogeneous terms of degree greater than or equal to zero. So, for $\mathbf{t}=(1,3)$, the diffeomorphism φ can be decomposed as $\varphi=\Psi\circ\phi$ with $\Psi(x,y)=(x,y+q_{20}x^2)^T$ and $\phi(x,y)=(x,b_{30}x^3+y)^T+(a_{20}x^2,b_{11}xy+b_{40}x^4)^T+\cdots$, i.e. we will only consider the changes of variables with $q_{20}=0$.

In what follows, we denote as ϕ_* and ϕ^* the push-forward and pull-back defined by the diffeomorphism ϕ, respectively, that is $\phi_*\mathbf{F}(\mathbf{x})=(D\phi(\mathbf{x}))^{-1}\mathbf{F}(\phi(\mathbf{x}))$, see [15].

Proposition 2.1. *Let $\mathbf{F}$ be a smooth (analytic) vector field with $\mathbf{F}(\mathbf{0})=\mathbf{0}$ and $\mathcal{J}_{\mathbf{t}}^r\mathbf{F}=\mathbf{F}_r\in\mathcal{Q}_r^{\mathbf{t}}$. If $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) conjugated, then there exists a smooth (analytic) diffeomorphism $\phi=\sum_{i\geqslant 0}\phi_i+\bar\phi$, with $D\phi(\mathbf{0})=I$, $\phi_i\in\mathcal{Q}_i^{\mathbf{t}}$ and $\bar\phi$ flat at origin ($\bar\phi\equiv 0$, if ϕ analytic) which transforms $\mathbf{F}$ into $\mathbf{F}_r$.*

Proof. By hypothesis, there exists a smooth (analytic) near-identity change of coordinates $\mathbf{x}=\varphi(\mathbf{y})=\mathbf{y}+\text{h.o.t.}$, that transforms $\mathbf{F}$ into $\mathbf{F}_r$. The diffeomorphism φ has the form $\varphi=\sum_{i\geqslant k}\varphi_i+\bar\varphi$ with k integer number non-positive, $\varphi_i\in\mathcal{Q}_i^{\mathbf{t}}$, $\bar\varphi$ flat at origin and $D\varphi(\mathbf{0})=I$. It is easy to prove that φ can be decomposed as $\varphi=\Psi\circ\phi$ with $\Psi=\sum_{i=k}^0\Psi_i$ (polynomial), $\phi=\sum_{i\geqslant 0}\phi_i+\bar\phi$ and $D\phi(\mathbf{0})=D\Psi(\mathbf{0})=I$, being Ψ_i and ϕ_i $\mathbf{t}$-homogeneous vector fields of degree i and $\bar\phi$ flat at origin.

On the one hand, ϕ transforms $\mathbf{F}$ into $\mathbf{F}_r+\mathbf{R}$ with $\mathbf{R}$ having $\mathbf{t}$-terms of order greater than r. On the other hand, $\Psi^{-1}=\sum_{i=k}^0\tilde\Psi_i$ is a polynomial, $\tilde\Psi_0(\mathbf{x})=\mathbf{x}$, which takes $\mathbf{F}_r$ to $\mathbf{F}_r+\mathbf{L}$ with $\mathbf{L}$ having $\mathbf{t}$-terms of order smaller than r. As $\phi=\Psi^{-1}\circ\varphi$, it has that

$$\mathbf{F}_r+\mathbf{L}=\left(\Psi^{-1}\circ\varphi\right)_*\mathbf{F}=\phi_*\mathbf{F}=\mathbf{F}_r+\mathbf{R},$$

thus, $\mathbf{R}\equiv\mathbf{L}\equiv 0$. So, ϕ transforms $\mathbf{F}$ into $\mathbf{F}_r$. □

Now, we prove that any smooth (analytic) perturbation of $\mathbf{D}_0$ with $\mathbf{t}$-homogeneous terms of degree greater than zero (i.e. a vector field whose Poincaré–Dulac normal form has not got any non-linear resonant terms because it has not got non-linear resonant terms) is smoothly (analytically) conjugated to $\mathbf{D}_0$.

Proposition 2.2. *Let $\mathbf{G}$ be a smooth (analytic) vector field, with $\mathcal{J}_{\mathbf{t}}^0\mathbf{G}=\mathbf{D}_0$. Then, $\mathbf{G}$ and $\mathbf{D}_0$ are smoothly (analytically) conjugated.*

Proof. First, we prove that $\mathbf{D}_0$ and $\mathcal{J}_{\mathbf{t}}^\infty\mathbf{G}$ are formally conjugated. To do that, we consider the linear map $\mathbf{L}_k:\mathcal{Q}_k^{\mathbf{t}}\to\mathcal{Q}_k^{\mathbf{t}}$ given by $\mathbf{L}_k(\mathbf{P}_k)=[\mathbf{P}_k,\mathbf{D}_0]$. As $\mathbf{L}_k(\mathbf{P}_k)=k\mathbf{P}_k$, it has that the range of the operator $\mathbf{L}_k$ is $\mathcal{Q}_k^{\mathbf{t}}$. Therefore, by normal form theory, it follows that there exists a formal near-identity nonlinear change of coordinates which transforms $\mathcal{J}_{\mathbf{t}}^\infty\mathbf{G}$ into $\mathbf{D}_0$, i.e. $\mathbf{D}_0$ and $\mathbf{G}$ are formally conjugated.

In Chen [10] is proved that two hyperbolic vector fields (a vector field such that the spectrum of its linear approximation does not intersect the imaginary axis) are smoothly conjugated if and only if they are formally conjugated. Thus, as $\mathbf{D}_0$ and any perturbation $\mathbf{G}$ of $\mathbf{D}_0$ are hyperbolic, it has that $\mathbf{G}$ and $\mathbf{D}_0$ are smoothly conjugated.

We prove that if $\mathbf{G}$ is analytic then $\mathbf{G}$ and $\mathbf{D}_0$ are analytically conjugated. If the eigenvalues of the matrix of the linear part of $\mathbf{G}$ at the origin are resonant, i.e. there is $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $\sum_{j=1}^n \alpha_j \geqslant 2$ such that $t_j = \alpha \cdot \mathbf{t} = \sum_{i=1}^n \alpha_i t_i$ for some $j \in \{1, \ldots, n\}$, it is known that there exists an analytic change of variables ϕ which transforms $\mathbf{G}$ into the Poincaré–Dulac normal form, that is $\mathbf{D}_0 + h_0$, where h_0 has only resonant terms, i.e. terms $\mathbf{x}^\alpha \mathbf{e}_j$ (see Arnold [5]), and as $\mathbf{x}^\alpha \in \mathcal{P}^{\mathbf{t}}_{\alpha\cdot\mathbf{t}}$ then $\mathbf{x}^\alpha \mathbf{e}_j \in \mathcal{Q}^{\mathbf{t}}_{\alpha\cdot\mathbf{t}-t_j} = \mathcal{Q}^{\mathbf{t}}_0$. From Proposition 2.1, the change of variables ϕ can be chosen of the form $\phi = \phi_0 + \cdots$ with $D\phi(\mathbf{0}) = D\phi_0(\mathbf{0}) = I$ and $\phi_* \mathbf{G} = \mathbf{D}_0 + h_0$, then $(\phi_0)_*(\mathcal{J}_{\mathbf{t}}^0 \mathbf{G}) = (\phi_0)_* \mathbf{D}_0 = \mathbf{D}_0 + h_0$. The change ϕ_0 is invertible and $\phi_0^{-1}(\mathbf{D}_0 + h_0) = \mathbf{D}_0$.

Therefore, $\mathbf{G}$ and $\mathbf{D}_0$ are analytically conjugated. If $\mathbf{t}$ is not resonant, the result is followed by applying the Poincaré Theorem since the eigenvalues of the linear approximation is in the Poincaré domain. □

Proof of Theorem 1.2.

Necessity. From Proposition 2.1, if $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) conjugated then there exists a smooth (analytic) ϕ with $D\phi(\mathbf{0}) = I$ which transforms the system $\mathbf{F}$ into $\mathbf{F}_r$, i.e. $\phi_* \mathbf{F} = \mathbf{F}_r$. Also, as $[\mathbf{F}_r, \mathbf{D}_0] = r\mathbf{F}_r$ then it has that

$$[\mathbf{F}, \phi^* \mathbf{D}_0] = [\phi^* \mathbf{F}_r, \phi^* \mathbf{D}_0] = \phi^* [\mathbf{F}_r, \mathbf{D}_0] = r\phi^* \mathbf{F}_r = r\mathbf{F},$$

with $\phi^* \mathbf{D}_0 = \mathbf{D}_0 + \cdots$. Thus, taking $\mathbf{G} = \phi^* \mathbf{D}_0$ it arrives at $[\mathbf{F}, \mathbf{G}] = r\mathbf{F}$.

Sufficiency. Let $\mathbf{G}$ be a smooth (analytic) vector field with $\mathcal{J}_{\mathbf{t}}^0 \mathbf{G} = \mathbf{D}_0$ and $[\mathbf{F}, \mathbf{G}] = r\mathbf{F}$. By the Propositions 2.1 and 2.2, we can assert that there is a smooth (analytic) change of variables $\mathbf{x} = \phi(\mathbf{y}) = \sum_{i\geqslant 0} \phi_i(\mathbf{y}) + \bar{\phi}$, with $D\phi(\mathbf{0}) = I$, $\phi_i \in \mathcal{Q}_i^{\mathbf{t}}$ and $\bar{\phi}$ flat at origin, such that $\phi_* \mathbf{G} = \mathbf{D}_0$. Let $\phi_* \mathbf{F} = \bar{\mathbf{F}} = \sum_{j\geqslant 0} \bar{\mathbf{F}}_{r+j} + \bar{f}$ with $\bar{\mathbf{F}}_r = \mathbf{F}_r$ and $\bar{f}$ flat at origin, then $[\bar{\mathbf{F}}, \mathbf{D}_0] = r\bar{\mathbf{F}}$, therefore

$$r\left(\sum_{j\geqslant 0} \bar{\mathbf{F}}_{r+j} + \bar{f}\right) = r\bar{\mathbf{F}} = [\bar{\mathbf{F}}, \mathbf{D}_0] = \sum_{j\geqslant 0} (r+j)\bar{\mathbf{F}}_{r+j} + [\bar{f}, \mathbf{D}_0].$$

So, it follows that $\bar{\mathbf{F}}_j \equiv 0$, for all $j > 0$, and $[\bar{f}, \mathbf{D}_0] = r\bar{f}$.

Let us prove that $\bar{f}$ is null. In fact, by doing the change of variables $\mathbf{x} = \xi(\rho, \bar{\mathbf{x}}) = (\rho^{t_1}\bar{x}_1, \ldots, \rho^{t_n}\bar{x}_n)^T$ with $\bar{\mathbf{x}}$ being onto the n-ellipsoid $\sum_{i=1}^n \bar{x}_i^{2L/t_i} = 1$, $L = \prod_{i=1}^n t_i$, the vector fields $\mathbf{D}_0$ and $\bar{f}$ are transformed into $\xi_* \mathbf{D}_0 = \rho\partial_\rho$ and $\xi_* \bar{f} = g_0(\rho, \bar{\mathbf{x}})\partial_\rho + \sum_{i=1}^n g_i(\rho, \bar{\mathbf{x}})\partial_{\bar{x}_i}$ with g_j, $j = 0, \ldots, n$, flat at $\rho = 0$. Imposing $[\xi_* \bar{f}, \xi_* \mathbf{D}_0] = r\xi_* \bar{f}$, it has that $(\rho\frac{\partial g_0}{\partial\rho} - g_0)\partial_\rho + \sum_{i=1}^n (\rho\frac{\partial g_i}{\partial\rho})\partial_{\bar{x}_i} = rg_0\partial_\rho + \sum_{i=1}^n rg_i\partial_{\bar{x}_i}$. Thus, $g_0(\rho, \bar{\mathbf{x}}) = \rho^{1+r}\bar{g}_0(\bar{\mathbf{x}})$ and $g_i(\rho, \bar{\mathbf{x}}) = \rho^r \bar{g}_i(\bar{\mathbf{x}})$, $i = 1, \ldots, n$. So, $g_0 \equiv 0$ and $g_i \equiv 0$, since they are flat at $\rho = 0$. Therefore, $\bar{f} \equiv 0$, i.e. $\phi_* \mathbf{F} = \mathbf{F}_r$.

The second part follows from the proof. □

Now, we give an auxiliary result.

Lemma 2. *Given a smooth (analytic) scalar function η, with $\eta(\mathbf{0})=0$, and a smooth (analytic) vector field $\mathbf{G}$ with $\mathcal{J}_{\mathbf{t}}^{0}\mathbf{G}=\mathbf{D}_0$, there exists a smooth (analytic) scalar function λ with $\lambda(\mathbf{0})=0$, such that*

$$L_{\mathbf{G}}\lambda(\mathbf{x})=\big(1+\lambda(\mathbf{x})\big)\eta(\mathbf{x}). \tag{1}$$

Proof. By Proposition 2.2, there exists a change of variables $\mathbf{x}=\phi(\mathbf{y})$, with ϕ smooth (analytic), which transforms $\mathbf{G}$ into $\mathbf{D}_0$. In these coordinates, Eq. (1) becomes

$$L_{\mathbf{D}_0}\bar{\lambda}(\mathbf{y})=\big(1+\bar{\lambda}(\mathbf{y})\big)\bar{\eta}(\mathbf{y})$$

with $\bar{\eta}(\mathbf{0})=0$, or equivalently, $L_{\mathbf{D}_0}\mathrm{Ln}(1+\bar{\lambda})(\mathbf{y})=\bar{\eta}(\mathbf{y})$. It is easy to check that the function

$$\bar{\lambda}(\mathbf{y})=\exp\left(\int_{-\infty}^{0}\bar{\eta}\big(e^{At}\mathbf{y}\big)\,dt\right)-1$$

with $A=\mathrm{diag}(t_1,\dots,t_n)$, satisfies the equation and $\bar{\lambda}(\mathbf{0})=0$. Moreover, $\bar{\lambda}$ and $\bar{\eta}$ have the same regularity. Undoing the change of variables, it gives the result. □

Proof of Theorem 1.3.

Necessity. From Proposition 2.1, if $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) orbital equivalent, there exists a diffeomorphism ϕ, and a parametrization by time $dt=\frac{d\tau}{1+f(\mathbf{y})}$ where f is a smooth (analytic) function with $f(\mathbf{0})=0$, such that $\phi_*\mathbf{F}=(1+f)\mathbf{F}_r$. So, we have that the vector field $\mathbf{F}$ is smoothly (analytically) conjugated to $(1+f)\mathbf{F}_r$.

Performing the inverse change ϕ^* to the vector field $\mathbf{F}_r$, this is transformed into $\tilde{\mathbf{F}}(\mathbf{x})$ with $\tilde{\mathbf{F}}=(1+g)\mathbf{F}$ and g smooth (analytic) function, $g(\mathbf{0})=0$. Thus, by Theorem 1.2, there exists a smooth (analytic) vector field $\mathbf{G}=\sum_{j\geqslant 0}\mathbf{G}_j$, $\mathbf{G}_j\in\mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{G}_0=\mathbf{D}_0$, such that $[(1+g)\mathbf{F},\mathbf{G}]=r(1+g)\mathbf{F}$. On the other hand, it has that

$$\big[(1+g)\mathbf{F},\mathbf{G}\big]=(1+g)[\mathbf{F},\mathbf{G}]+(L_{\mathbf{G}}g)\mathbf{F}.$$

Therefore, $[\mathbf{F},\mathbf{G}]=(r-\frac{L_{\mathbf{G}}g}{1+g})\mathbf{F}$. Taking $\mu=(r-\frac{L_{\mathbf{G}}g}{1+g})$, the result follows.

Sufficiency. We assume that there exists a smooth (analytic) vector field $\mathbf{G}$, with $\mathcal{J}_{\mathbf{t}}^{0}\mathbf{G}=\mathbf{D}_0$, verifying $[\mathbf{F},\mathbf{G}]=\mu\mathbf{F}$ with μ a smooth (analytic) function, $\mu(\mathbf{0})=r$. Let λ be a smooth (analytic) scalar function with $\lambda(\mathbf{0})=0$ such that

$$L_{\mathbf{G}}\lambda=(1+\lambda)(r-\mu)$$

which exists by Lemma 2 for $\eta=r-\mu$. Then, we have that

$$\begin{aligned}\big[(1+\lambda)\mathbf{F},\mathbf{G}\big]&=(1+\lambda)[\mathbf{F},\mathbf{G}]+L_{\mathbf{G}}\lambda\mathbf{F}\\&=(1+\lambda)[\mathbf{F},\mathbf{G}]+(1+\lambda)(r-\mu)\mathbf{F}\\&=(1+\lambda)\big([\mathbf{F},\mathbf{G}]+r\mathbf{F}-\mu\mathbf{F}\big)=r(1+\lambda)\mathbf{F}.\end{aligned}$$

By Theorem 1.2, $(1+\lambda)\mathbf{F}$ is smoothly (analytically) conjugated to $\mathbf{F}_r$ and thus $\mathbf{F}$ and $\mathbf{F}_r$ are smoothly (analytically) orbital equivalent. □

3. Examples

In this section, we show some very simple examples. The first four examples are nilpotent vector fields, three of them are two-dimensional and the fourth is a three-dimensional system. The fifth example is a system with linear part null, the so-called generalized nilpotent system.

Example 1. Let us consider the nilpotent system

$$\begin{aligned}\dot{x} &= y + Ax^2 - 2y^2 - Ax^2y + y^3 + Bx^4,\\ \dot{y} &= -Bx^3 + Bx^3y,\end{aligned} \tag{2}$$

with $B \neq 0$.

The $(1,2)$-homogeneous expansion of the vector field associated to (2) is given by $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_3 + \mathbf{F}_5$ where

$$\begin{aligned}\mathbf{F}_1 &= \left(y + Ax^2, -Bx^3\right)^T,\\ \mathbf{F}_3 &= \left(-2y^2 - Ax^2y + Bx^4, Bx^3y\right)^T,\\ \mathbf{F}_5 &= \left(y^3, 0\right)^T.\end{aligned}$$

One can check that

$$\left[\mathbf{F}(x,y), \left(x - 2xy, 2y - 2y^2\right)^T\right] = (1 - 4y)\mathbf{F}(x,y).$$

Therefore, system (2) is orbitally equivalent to the system

$$(\dot{x}, \dot{y})^T = \left(y + Ax^2, -Bx^3\right)^T. \tag{3}$$

We now analyze system (3). The change of variables $x_1 = x$, $x_2 = y + \frac{1}{2}Ax^2$, transforms (3) into

$$\dot{x}_1 = x_2 + \frac{A}{2}x_1^2, \qquad \dot{x}_2 = -\left(B - \frac{A^2}{2}\right)x_1^3 + Ax_1x_2.$$

Next, we perform the change to generalized polar coordinates

$$x_1 = uCs(\theta), \qquad x_2 = u^2Sn(\theta), \qquad dt = -\frac{4}{u}\,d\tau,$$

where $(Cs(\theta), Sn(\theta))$ is the unique solution to the Cauchy problem

$$\frac{dx}{d\theta} = -2y, \qquad \frac{dy}{d\theta} = 4x^3,$$

with $x(0) = 1$, $y(0) = 0$. So, system (3) becomes

$$\begin{aligned}u' &= -2uCs(\theta)\left[A - \left(B - \frac{A^2}{2} - 2\right)Cs^2(\theta)Sn(\theta)\right],\\ \theta' &= 2Sn^2(\theta) + \left(B - \frac{A^2}{2}\right)Cs^4(\theta),\end{aligned} \tag{4}$$

where $'$ represents $\frac{d}{d\tau}$. Thus, if $A^2 - 2B < 0$, the origin of (3) is a monodromic point. Moreover, system (3) is time reversible under the change of variables $(x,y,t) \to (-x,y,-t)$, therefore the origin is a center. If $A^2 - 2B > 0$, system (4) has two hyperbolic equilibriums, hence the origin of (3) is a saddle point. And if $A^2 - 2B = 0$, the origin of (3) is a degenerate point.

Example 2. Let us consider the nilpotent vector field

$$\mathbf{F}(x,y)=\big(y-ax^2,x^2+2axy-2a^2x^3\big)^T.$$

This vector field, for $\mathbf{t}=(2,3)$, is expressed as $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3$ where $\mathbf{F}_1(x,y)=(y,x^2)^T$ is its $(2,3)$-homogeneous part of minimum degree and

$$\mathbf{F}_2(x,y)=a\big(-x^2,2xy\big)^T,\qquad \mathbf{F}_3(x,y)=-2a^2\big(0,x^3\big)^T.$$

It can be checked that $\mathbf{G}(x,y)=(2x,3y+ax^2)^T$, which has a linear part given by $(2x,3y)^T$, verifies $[\mathbf{F},\mathbf{G}]=\mathbf{F}$ and thus we conclude that $\mathbf{F}$ can be reduced to $(y,x^2)^T$ by a convergent transformation.

Example 3. Let us consider the nilpotent vector field

$$\mathbf{F}(x,y)=\begin{pmatrix} y(1+y)^4+(1-c)x^3(1+cy)^2 \\ x^2(1+y)(1+cy)^3 \end{pmatrix}, \tag{5}$$

whose $(2,3)$-homogeneous expansion is of the form $(y,x^2)^T+\cdots$. For

$$\mathbf{G}(x,y)=\left(\frac{(2+(5-c)y+2cy^2)x}{(1+y)(1+cy)},3y\right)^T,\qquad \mu(x,y)=\frac{1+4cy+10y+13cy^2}{(1+y)(1+cy)}$$

it has $[\mathbf{F}(x,y),\mathbf{G}(x,y)]=\mu(x,y)\mathbf{F}(x,y)$, with $\mathbf{G}(x,y)=(2x,3y)^T+\cdots$ and $\mu(\mathbf{0})=1$, thus $\mathbf{F}$ is orbitally equivalent to $(y,x^2)^T$.

Also, it has that $\mathbf{F}$ is conjugated to $(1+\lambda(x,y))(y,x^2)^T$ where $1+\lambda(x,y)=\frac{1}{(1+y)(1+cy)}$.

Example 4. Let us consider the three-dimensional system

$$\begin{aligned}
\dot{x}&=y+2zT_1(x,y,z)+2czT_2(x,y,z),\\
\dot{y}&=z-y^2+c\big(x-z^2\big)^2,\\
\dot{z}&=T_1(x,y,z)+cT_2(x,y,z),
\end{aligned} \tag{6}$$

with $T_1(x,y,z)=(x-z^2)^2+2y(z-y^2)$ and $T_2(x,y,z)=2y(x-z^2)(x-z^2-1)$.

For $\mathbf{t}=(3,4,5)$, the vector field associated to (6) has the form $\mathbf{F}(x,y,z)=(y,z,x^2)^T+\tilde{\mathbf{F}}$, where $\tilde{\mathbf{F}}$ has $(3,4,5)$-homogeneous parts of a higher degree. This vector field is conjugated to $(y,z,x^2)^T$, since $[\mathbf{F},\mathbf{G}]=\mathbf{F}$, where

$$\mathbf{G}(x,y,z)=\big(3x+z\big(7z+6y^2\big)-2cz\big(x-z^2\big)^2,4y,5z+3y^2-c\big(x-z^2\big)^2\big)^T.$$

Example 5. Let us consider the family of vector fields

$$\mathbf{F}(x,y)=\big(y^3+P_5(x,y),Q_5(x,y)\big)^T, \tag{7}$$

where P_5 and Q_5 are quintic homogeneous polynomials with $Q_5(1,0)\neq 0$. Integrability and centers of these vector fields have been studied in Giné [11]. The $(2,3)$-homogeneous expansion of the vector fields of the family is given by $\mathbf{F}=\mathbf{F}_7+\cdots+\mathbf{F}_{13}$ where

$$\begin{aligned}
&\mathbf{F}_7=\big(y^3,cx^5\big)^T,\qquad \mathbf{F}_8=\big(a_{50}x^5,b_{41}x^4y\big)^T,\qquad \mathbf{F}_9=\big(a_{41}x^4y,b_{32}x^3y^2\big)^T,\\
&\mathbf{F}_{10}=\big(a_{32}x^3y^2,b_{23}x^2y^3\big)^T,\qquad \mathbf{F}_{11}=\big(a_{23}x^2y^3,b_{14}xy^4\big)^T,\\
&\mathbf{F}_{12}=\big(a_{14}xy^4,b_{05}y^5\big)^T,\qquad \mathbf{F}_{13}=\big(a_{05}y^5,0\big)^T.
\end{aligned}$$

We look for systems of the family (7) which, by means of a change of variables, can be transformed into $(y^3, cx^5)^T$, $c \neq 0$. It has the following result.

Proposition 3.1. *No vector field of the family* (7) *is neither smoothly conjugated, nor formally conjugated, to* $(y^3, cx^5)^T$, $c \neq 0$, *except for itself.*

Proof. As $(y^3, cx^5)^T$ is an integrable vector field, $\frac{1}{4}y^4 - \frac{c}{6}x^6$ is an analytic first integral, the vector fields of the family (7) which could be smoothly conjugated to $(y^3, cx^5)^T$ should also be integrable. By Giné [11], these are

(Fam1) $Q_5 \equiv 0$,
(Fam2) Hamiltonian system,
(Fam3) $P_5(x, y) = y^3(1 + a_{23}x^2)$, $Q_5(x, y) = b_{50}x^5 + b_{14}xy^4$.

It is easy to check that there isn't any change in the form $x = x + \cdots,\ \ y = y + \cdots$ which transforms a vector field of (Fam1) into $(y^3, cx^5)^T$, $c \neq 0$.

If (7) is of (Fam2) then

$$b_{41} = a_{50} = 2a_{41} + b_{32} = a_{32} + b_{23} = a_{23} + 2b_{14} = a_{14} + 5b_{05} = 0.$$

By imposing the existence of a vector field $\mathbf{G}(x, y) = (2x, 3y)^T + \mathbf{G}_1 + \cdots$, such that $[\mathbf{F}, \mathbf{G}] = 7\mathbf{F}$, we obtain recursively the conditions $a_{41} = 0$, $a_{23} = 0$, $a_{14} = 0$, $3ca_{05} + a_{32}^2 = 0$ and $a_{32} = 0$, that is, we get $(y^3, cx^5)^T$.

Analogously, in the case (Fam3) we have the conditions $6b_{14} + 5a_{23} = 0$, $a_{23} = 0$. So, the result follows. □

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